SINGULAR CARDINALS AND SQUARE PROPERTIES

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ABSTRACT. We analyze the effect of singularizing cardinals on square properties. By work of Džamonja-Shelah and of Gitik, if you singularize an inaccessible cardinal to countable cofinality while preserving its successor, then $\square_{\kappa,\omega}$ holds in the bigger model. We extend this to the situation where every regular cardinal in an interval $[\kappa,\nu]$ is singularized, for some regular cardinal ν. More precisely, we show that if $V \subset W$, $\kappa < \nu$ are cardinals, where ν is regular in V, κ is a singular cardinal in W of countable cofinality, $\mathrm{cf}^W(\tau) = \omega$ for all V-regular $\kappa \leq \tau \leq \nu$, and $(\nu^+)^V = (\kappa^+)^W$, then $W \models \square_{\kappa,\omega}$.

1. INTRODUCTION

The square principle was isolated by Jensen in his fine structure analysis of L. Square at κ, $\square_\kappa$, states that there exists a sequence $\langle C_\alpha \mid \alpha < \kappa^+\rangle$ such that each C_α is a club subset of α, $o.t.(C_\alpha) \leq \kappa$, and if $\delta \in \lim C_\alpha$, then $C_\alpha \cap \delta = C_\delta$. There are various weakenings allowing multiple guesses for the clubs at each point. More precisely, the principle $\square_{\kappa,\lambda}$ states that there is a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \kappa^+\rangle$, such that:

(1) $1 \leq |\mathcal{C}_\alpha| \leq \lambda$,
(2) every $C \in \mathcal{C}_\alpha$ is a club in α, $o.t.(C) \leq \kappa$,
(3) if $C \in \mathcal{C}_\alpha$ and $\delta \in \lim(C)$, then $C \cap \delta \in \mathcal{C}_\delta$.

Square principles hold in models that sufficiently resemble L. On the other hand they are at odds with reflection properties and fail above large cardinals. For example, there is tension between failure of SCH and failure of the weaker square properties. One reason is that it is difficult to avoid weaker square principles at singular cardinals. The standard way of singularizing cardinals is by Prikry forcing, and Cummings and Schimmerling showed in [2] that after Prikry forcing at κ, $\square_{\kappa,\omega}$ holds in the generic extension. More generally, by arguments in Gitik [4] and independently in Džamonja-Shelah [3], it was implicit that if $V \subset W$ are transitive class models of ZFC such that κ is an inaccessible cardinal in V, singular of countable cofinality in W, and $(\kappa^+)^V = (\kappa^+)^W$, then $W \models \square_{\kappa,\omega}$. On the other hand in Gitik-Sharon [5], a forcing was constructed where κ is singularized, $(\kappa^{+\omega+1})^V$ becomes the successor of κ, and weak square of κ fails in the generic extension.

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The natural question is what happens if κ is singularized and an interval of cardinals above κ is collapsed. In this paper we show that if $V \subset W$ are such that κ is a cardinal in V and W, $\mathrm{cf}^W(\kappa) = \omega$, $\nu > \kappa$ is a regular cardinal, every V-regular cardinal τ, $\kappa \leq \tau \leq \nu$ has cofinality ω in W, and $(\nu^+)^V = (\kappa^+)^W$, then $\square_{\kappa,\omega}$ holds in the outer model. Similarly to what was done in [3] and [4], we show the existence of a "pseudo Prikry sequence" witnessing that ν is singular in the bigger model. Moreover, for $\nu = \kappa^{+n}$ for some n, we generalize these arguments to obtain a pseudo Prikry sequence in W through $\mathcal{P}_\kappa((\kappa^{+n})^V)$. We use the term "pseudo Prikry" because the properties of these sequences will be the best approximation of the genericity of a Prikry sequence, without referring to any measure.

We will use the following notation: $\mathrm{cof}(\tau)$ and $\mathrm{cof}(< \tau)$ denote points of cofinality τ and less than τ, respectively. Similarly, we write $\mathrm{cof}^W(\tau)$ to denote the points of cofinality τ in W. For a set C, $\lim(C)$ is the set of all limit points of C and $\mathrm{nacc}(C)$ is the non-limit points of C.

2. Pseudo Prikry sequences

We start with a fact due to Shelah, whose proof can be found in [6], Theorem 2.14.

Proposition 2.1. *Suppose that $\nu > \kappa > \aleph_1$ are cardinals, where ν is regular. Then there is a sequence $\langle S_\delta \mid \delta < \nu\rangle$ of stationary subsets of ν^+, such that $\bigcup_{\delta<\nu} S_\delta = \nu^+ \cap \mathrm{cof}(< \kappa)$ and each S_δ carries a partial square sequence $\langle C^\delta_\alpha \mid \alpha \in S_\delta \cap Lim\rangle$. More precisely:*

- *each C^δ_α is a club subset of α with o.t.$(C^\delta_\alpha) < \kappa$, and*
- *if $\beta \in \lim(C^\delta_\alpha)$, then $\beta \in S_\delta$ and $C^\delta_\alpha \cap \beta = C^\delta_\beta$.*

Theorem 2.2. *Suppose $V \subset W$ are such that $\kappa < \nu$ are cardinals in V, ν is regular in V, $(\omega_1)^W < \kappa$, $\mathrm{cf}^W(\kappa) = \omega$, for all V-regular cardinals, τ, $\kappa \leq \tau \leq \nu$, $\mathrm{cf}^W \tau = \omega$, and $(\nu^+)^V = (\kappa^+)^W$. Suppose that $\langle D_\alpha \mid \alpha < (\nu^+)^V\rangle \in V$ is a sequence of club subsets of ν. Then in W there is a sequence $\langle \delta_n \mid n < \omega\rangle$ cofinal in ν, such that for every $\alpha < (\nu^+)^V$, for all large n, $\delta_n \in D_\alpha$. Furthermore, for any $\lambda < \kappa$, such that λ is regular in W, we may assume that for all n, $\mathrm{cf}^W(\delta_n) \geq \lambda$.*

Proof. Note that we are not assuming that κ remains a cardinal in W. Denote $\mu := \nu^+$, and let $\langle D_\alpha \mid \alpha < \mu\rangle \in V$ be club subsets of ν. First we will construct a sequence $\langle \delta_n \mid n < \omega\rangle$ meeting each of these sets on a final segment. Then we will amend the argument to ensure that the cofinality is as desired.

In V, let $\langle S_\delta \mid \delta < \nu\rangle$ be as in Proposition 2.1. In particular, each S_δ carries a partial square sequence of clubs of order type less than κ. Since $\bigcup_{\delta<\nu} S_\delta = \mu \cap \mathrm{cof}^V(< \kappa) \supset \mu \cap \mathrm{cof}^W(\omega_1)$, let $\delta < \nu$ be such that in W, $S_\delta \cap \mathrm{cof}^W(\omega_1)$ is stationary, and denote $S := S_\delta$. Let $\langle C_\alpha \mid \alpha \in S\rangle$ be the corresponding partial square sequence.

Claim 2.3. *In V there is a sequence $\langle D^*_\alpha \mid \alpha < \mu\rangle$ of club subsets of ν, such that*

1. *For each $\alpha < \mu$, $D^*_\alpha \subset D_\alpha$,*
2. *For every $\alpha < \beta < \mu$, $|D^*_\beta \setminus D^*_\alpha| < \nu$*
3. *For every $\alpha < \beta < \mu$, $\beta \in S$, if $\alpha \in C_\beta$, then $D^*_\beta \subset D^*_\alpha$.*

Proof. Work in V. Let $D^*_0 = D_0$. Suppose we have defined $\langle D^*_\alpha \mid \alpha < \beta\rangle$, $\beta < \mu$. If $\beta \in S$, set

$$D^*_\beta = D_\beta \cap \triangle_{\alpha<\beta} D^*_\alpha \cap \bigcap_{\alpha \in C_\beta} D^*_\alpha.$$

If $\beta \notin S$, let $D^*_\beta = D_\beta \cap \triangle_{\alpha<\beta} D^*_\alpha$.

□

In W, let $\langle \lambda_n \mid n < \omega\rangle$ be an increasing cofinal sequence in ν. Let I_n be the interval $(\lambda_n, \lambda_{n+1})$. For every $\alpha < \mu$, let $d_\alpha = \{n \mid D^*_\alpha \cap I_n \neq \emptyset\}$, and define a sequence $\langle \lambda^\alpha_n \mid n \in d_\alpha\rangle$, by setting

$$\lambda^\alpha_n = \sup(D^*_\alpha \cap I_n).$$

Note that for $\alpha < \beta$, $d_\beta \setminus d_\alpha$ is finite, and if $\beta \in S, \alpha \in C_\beta$, then $d_\beta \subset d_\alpha$. Also, it follows that for all large n, if λ^β_n is defined, then so is λ^α_n and $\lambda^\beta_n \leq \lambda^\alpha_n$.

Denote $E_\alpha := \langle \lambda^\alpha_n \mid n \in d_\alpha\rangle$. For $\alpha < \beta$, we say that there is *a major change between α and β* if $E_\alpha \triangle E_\beta$ is infinite. We claim that for $\alpha < \beta < \gamma$, if there is a major change between α and β, then there is a major change between α and γ. For otherwise, if $E_\alpha \triangle E_\gamma$ is finite, then for all large n, $\lambda^\alpha_n = \lambda^\gamma_n$, when defined. So, for all large n, $\lambda^\gamma_n \leq \lambda^\beta_n \leq \lambda^\alpha_n = \lambda^\gamma_n$, which means that $E_\alpha \triangle E_\beta$ must have been finite as well.

Claim 2.4. *There is some α, such that for every $\alpha < \beta < \mu$, there is no major change between α and β.*

Proof. Otherwise, for every α, let $f(\alpha) > \alpha$ be such that there is a major change between α and $f(\alpha)$. The set $C := \{\alpha < \mu \mid (\forall \beta < \alpha)(f(\beta) < \alpha)\}$ is club, so pick $\alpha \in \lim(C) \cap S \cap \mathrm{cof}^W(\omega_1)$. Let $\langle \alpha_i \mid i < \omega_1\rangle$ be a cofinal sequence through $\lim(C_\alpha) \cap C$. Then if $i < j < \omega_1$, $D^*_{\alpha_j} \subset D^*_{\alpha_i}$, and so $d_{\alpha_j} \subset d_{\alpha_i}$. Then $\langle d_{\alpha_i} \mid i < \omega_1\rangle$ must be eventually constant. So, let d be such that for all large i, $d = d_{\alpha_i}$.

Now for every $n \in d$, the sequence $\langle \lambda^{\alpha_i}_n \mid i < \omega_1\rangle$ is weakly decreasing, so for some $i_n < \omega_1$ and $\bar\lambda_n$, we have that for all $i \geq i_n$, $\lambda^{\alpha_i}_n = \bar\lambda_n$. Set $i = \sup_{n\in d} i_n < \omega_1$. Let $E := \{\bar\lambda_n \mid n \in d\}$. Then for all $\xi \geq i$, $E_{\alpha_\xi} = E$.

But for $i < \xi < \xi'$, since $\alpha_{\xi'} \in C$, we have that $f(\alpha_\xi) < \alpha_{\xi'}$, so there is a major change between α_ξ and $\alpha_{\xi'}$. But $E_{\alpha_\xi} = E = E_{\alpha_{\xi'}}$. Contradiction

□

Now let α be as in the claim and set $E := E_\alpha$. Then E is the desired sequence, i.e. for all $\beta < \mu$, on a final segment E is contained in D^*_β.

Next, we show how to get a sequence $\langle \delta_n \mid n < \omega \rangle$ as obtained above, but with the additional requirement that for all n, $\mathrm{cf}(\delta_n) \geq \lambda$ (in W). We follow the argument in Džamonja-Shelah [3]. First note that for any club $D \subset \nu$, the set

$$\{\delta \in D \mid o.t.(D \cap \delta) \text{ is divisible by } \lambda\}$$

is club, such that if δ is a non-limit point in that set, then $\mathrm{cf}^W(\delta) \geq \lambda$.

Fix a sequence $\langle a_\delta \mid \delta < \nu \rangle \in V$, such that each a_δ is a club subset of δ with order type $\mathrm{cf}^V(\delta)$. We define a sequence of clubs $\langle D^{**}_\alpha \mid \alpha < \mu \rangle$ by induction on α. Set $D^{**}_0 = \{\delta \in D_0 \mid o.t.(D_0 \cap \delta) \text{ is divisible by } \lambda\}$. Suppose we have already defined $\langle D^{**}_\alpha \mid \alpha < \beta \rangle$. If $\beta \in S$, let $D'_\beta = D_\beta \cap \triangle_{\alpha<\beta} D^{**}_\alpha \cap \bigcap_{\alpha \in C_\beta} D^{**}_\alpha$. If $\beta \notin S$, let $D'_\beta = D_\beta \cap \triangle_{\alpha<\beta} D^{**}_\alpha$. Then set $D^{**}_\beta := \{\delta \in D'_\beta \mid o.t.(D'_\beta \cap \delta) \text{ is divisible by } \lambda\}$.

Then $\langle D^{**}_\alpha \mid \alpha < \mu \rangle$ satisfy (1), (2), and (3) in Claim 2.3, with the new key property that if $\delta \in \mathrm{nacc}(D^{**}_\alpha)$, then $\mathrm{cf}^W(\delta) \geq \lambda$.

Now, for $\alpha < \mu$, let $T^\alpha_0 := E_\alpha$, where E_α is defined above, but with respect to D^{**}_α, and let

$$T^\alpha_{n+1} := T^\alpha_n \cup \{\sup(\rho \cap D^{**}_\alpha) \mid$$

$$(\exists \delta \in T^\alpha_n \cap \mathrm{cof}^W(<\lambda))\rho \in a_\delta, \rho > \sup(D^{**}_\alpha \cap \rho \cap T^\alpha_n)\}.$$

Set $T^\alpha = \bigcup_n T^\alpha_n$. Note that $T^\alpha \subset D^{**}_\alpha$.

Claim 2.5. *If $\delta \in nacc(T^\alpha)$, then* $\mathrm{cf}^W(\delta) \geq \lambda$.

Proof. Otherwise, suppose $\delta \in \mathrm{nacc}(T^\alpha)$ and $\mathrm{cf}^W(\delta) < \lambda$. Then, by definition of D^{**}_α, $\delta \notin \mathrm{nacc}(D^{**}_\alpha)$ i.e. $\delta \in \lim(D^{**}_\alpha)$. So $\delta = \sup(\{\sup(\rho \cap D^{**}_\alpha) \mid \rho \in a_\delta\})$.

Since $\delta \in \mathrm{nacc}(T^\alpha)$, $a_\delta \cap T^\alpha$ must be bounded in δ. Let $\bar{\delta} < \delta$ be such that $T^\alpha \cap (\delta \setminus \bar{\delta}) = \emptyset$. Then for all $\rho \in a_\delta \setminus \bar{\delta}$, $\rho > \sup(D^{**}_\alpha \cap \rho \cap T^\alpha_n)$. Now let n be such that $\delta \in T^\alpha_n$. Then since $\mathrm{cf}^W(\delta) < \lambda$, $\{\sup(\rho \cap D^{**}_\alpha) \mid \rho \in a_\delta \setminus \bar{\delta}\} \subset T^\alpha_{n+1}$. So $\sup(a_\delta \cap D^{**}_\alpha) < \delta$. Then $\delta \in \mathrm{nacc}(D^{**}_\alpha)$, and so $\mathrm{cf}^W(\delta) \geq \lambda$. $\square$

We say that there is *a major change* between α and β if $T^\alpha \triangle T^\beta$ is unbounded in ν. By running the same argument as above we get that there is some α, such that for all $\alpha < \beta < \mu$, there is no major change between α, β. The only difference here is that we take S, such that $S \cap \mathrm{cof}^W(\lambda)$ is stationary. Then setting $T := T^\alpha$, we have that for all $\beta < \mu$, for some δ, $T \setminus \delta \subset D^{**}_\beta$.

Then let $\langle \delta_n \mid n < \omega \rangle$ be cofinal in $\mathrm{nacc}(T)$. This sequence is as desired. $\square$

Next we generalize to club subsets of $\mathcal{P}_\kappa(\kappa^{+m})$. In the arguments below, for a set of ordinals y we define *the characteristic function of* y, χ_y to be a function defined on the regular cardinals $\eta \leq \sup(y)$ where $\chi_y(\eta) = \sup(y \cap \eta)$. We will need the following claim.

Claim 2.6. *Let κ be a regular cardinal, m a natural number. Then there is a club D of $P_\kappa(\kappa^{+m})$ such that for all $x, y \in D$ if for all $i \leq m$, $\chi_x(\kappa^{+i}) = \chi_y(\kappa^{+i})$ and $\mathrm{cf}(\chi_x(\kappa^{+i}))$ is uncountable, then $x = y$.*

Proof. Let χ be some big enough cardinal, and $<_\chi$ be a well-ordering of H_χ. Let $D' = \{M \cap \kappa^{+m} \mid M \prec \langle H_\chi, \in, <_\chi \ldots\rangle, |M| < \kappa, (\forall i < m)\kappa^{+i} \in M, M \cap \kappa \in \kappa\}$. Then $D' \cap \{x \mid (\forall i \leq m)(\mathrm{cf}(\chi_x(\kappa^{+i})) > \omega) \to x \text{ is } \omega\text{-closed}\}$ contains a club, so let D be a club contained in it.

Suppose $x, y \in D$ are such that for all $i \leq m$, $\chi_x(\kappa^{+i}) = \chi_y(\kappa^{+i})$ and $\mathrm{cf}(\chi_x(\kappa^{+i}))$ is uncountable. Since x, y are both ω-club in $\alpha := \sup(x)$, $x \cap y$ is also ω-club in α. Now let M, N witness that $x, y \in D$. By induction on i, we show that $x \cap \kappa^{+i} \subseteq y \cap \kappa^{+i}$. For $i = 0$, this is is given. Fix $i > 0$. For every $\rho \in x \cap y \cap \kappa^{+i}$, with $|\rho| = \kappa^{+i-1}$, by elementarity the $<_\chi$-least bijection from κ^{+i-1} to ρ is in $M \cap N$. Since by induction $M \cap \kappa^{+i-1} = N \cap \kappa^{+i-1}$, we get that $M \cap \rho \subseteq N \cap \rho$. By doing this for all $\rho \in x \cap y \cap \kappa^{+i}$, we get $x \cap \kappa^{+i} \subseteq y \cap \kappa^{+i}$. Taking $i = m$, we get $x \subseteq y$.

By an identical argument $y \subseteq x$, and so they are equal. □

Below $V \subseteq W$ are two transitive models of set theory. When we use some set theoretic terminology or notation like "a regular cardinal", δ^+ etc. we shall mean it in the sense of V, unless otherwise stated.

Theorem 2.7. *Suppose that κ is a regular cardinal in V, and $m < \omega$ is such that in W $\kappa, \kappa^+, \ldots \kappa^{+m}$ all have cofinality ω, $(\omega_1)^W < \kappa$, and $(\kappa^+)^W = \kappa^{+m+1}$. In V let $\langle D_\alpha \mid \alpha < \kappa^{+m+1}\rangle$ be a sequence of clubs of $P_\kappa(\kappa^{+m})$. Then in W there is a sequence $\langle x_n \mid n < \omega\rangle$ of elements of $P_\kappa(\kappa^{+m})$, such that for every $\alpha < \kappa^{+m+1}$ for large enough $n < \omega$, $x_n \in D_\alpha$.*

Moreover, if $\lambda < \kappa$ is an uncountable regular cardinal in W, we can assume that for $n < \omega$ and $k \leq m$, $\mathrm{cf}^W(\chi_{x_n}(\kappa^{+k})) \geq \lambda$.

Proof. We shall prove the theorem by induction on m. The case $m = 0$ follows from Theorem 2.2. Suppose the theorem is true for m. In order to prove the theorem for $m + 1$ we shall use the theorem for m and $\eta := \kappa^+$ in place of κ. Set $\delta = \kappa^{+m+1} = \eta^{+m}$ and $\mu = \delta^+$. Using Claim 2.1, fix a stationary set $S \subseteq \mu \cap \mathrm{cof}^W(\leq \lambda)$ such that there is a partial square sequence on $S, \langle C_\alpha \mid \alpha \in S\rangle$, and such that $S \cap \mathrm{cof}^W(\lambda)$ is stationary. As usual we can assume that the order type of C_α for $\alpha \in S$ is $\leq \lambda$.

Let $\langle D_\alpha \mid \alpha < \mu\rangle$ be a sequence of clubs in $P_\kappa(\delta)$. We can assume without loss of generality that for all $\alpha < \mu$, $D_\alpha \subseteq D$, where D is the club defined in Claim 2.6.

Claim 2.8. *In V there is a sequence $\langle D'_\alpha \mid \alpha < \mu\rangle$ of club subsets of $\mathcal{P}_\kappa(\delta)$, such that:*

1. *For each $\alpha < \mu$, $D'_\alpha \subset D_\alpha$,*
2. *For all $\alpha < \beta < \mu$, there is $\gamma < \delta$, such that $D'_\beta \setminus D'_\alpha \subset \{x \mid \gamma \notin x\}$*
3. *For all $\alpha < \beta < \mu$, with $\beta \in S$, if $\alpha \in C_\beta$, then $D'_\beta \subset D'_\alpha$.*

Proof. Work in V. For every $\beta < \mu$, fix a one-to-one function $f_\beta : \beta \to \delta$. Let $D'_0 = D_0$. Suppose we have defined $\langle D'_\alpha \mid \alpha < \beta \rangle$. If $\beta \in S$, set

$$D'_\beta = D_\beta \cap \{x \mid x \in \bigcap_{\gamma \in x} D'_{f_\beta^{-1}(\gamma)}\} \cap \bigcap_{\alpha \in C_\beta} D'_\alpha.$$

Otherwise, let $D'_\beta = D_\beta \cap \{x \mid x \in \bigcap_{\gamma \in x} D'_{f_\beta^{-1}(\gamma)}\}$.
$\langle D'_\alpha \mid \alpha < \mu \rangle$ is as desired. The γ in item (2) is $\gamma = f_\beta(\alpha)$. □

By applying the above claim and shrinking the D_α's if necessary, we may assume they satisfy the conclusion of the claim. For $\alpha < \mu$, fix an algebra with λ many operations on δ, $\mathcal{A}_\alpha$, such that every $x \in P_\kappa(\delta)$ which is a subalgebra of $\mathcal{A}_\alpha$ is in D_α. By our assumptions about D_α, we can arrange that for $\alpha < \beta$ there is $\rho < \delta$ such that a subalgebra of $\mathcal{A}_\beta$ containing ρ is also closed under the operations of $\mathcal{A}_\alpha$. Also we can arrange that if α is a limit point of C_β, then a subalgebra of $\mathcal{A}_\beta$ is closed under the operations of $\mathcal{A}_\alpha$. (The last clause is the one that requires the algebras to have λ many operations.)

For $\alpha < \mu$, let D^*_α be the club in $P_\eta(\delta)$ defined by being a subalgebra of $\mathcal{A}_\alpha$. By the induction assumption for m, η and $\lambda \geq (\omega_1)^W$ and the sequence of clubs $\langle D^*_\alpha \mid \alpha < \mu \rangle$, we get a sequence $\langle z_k \mid k < \omega \rangle$, such that for every $\alpha < \mu$ for large enough k, $z_k \in D^*_\alpha$. Without loss of generality, we can assume that $\langle z_k \mid k < \omega \rangle$ is $\subseteq$ increasing and $\kappa \subseteq z_0$. Also we choose the z_k's, so that the characteristic function $\chi_{z_k}(\kappa^{+i})$ has W-cofinality at least λ.

Since for $i \leq m+1$, $\mathrm{cf}^W(\kappa^{+i}) = \omega$, we fix in W a $\subseteq$-increasing sequence $\langle t_n \mid n < \omega \rangle$ of elements of $P_\kappa(\delta)$ such that $\cup_{n<\omega} t_n = \delta$ and for all $n < \omega$ $t_n \subseteq z_n$.

Below, we will construct the desired sequence $\langle x_n \mid n < \omega \rangle$ to be such that $t_n \subseteq x_n \subseteq z_n$ and for $0 < i \leq m+1$ $\chi_{x_n}(\kappa^{+i}) = \chi_{z_n}(\kappa^{+i})$. Note that since $\mathrm{cf}^W(\chi_{z_n}(\kappa^{+i})) \geq \lambda$, we have that the cofinality of this ordinal in V is uncountable and below κ.

For $\alpha < \mu, n < \omega$, in V, define E^n_α to be the set of $\xi < \kappa$, such that there exists $y \in P_\kappa(\delta)$, such that:

- y is closed under the operations of $\mathcal{A}_\alpha$,
- $y \cap \kappa = \xi$,
- for $0 < i \leq m+1$, $\chi_y(\kappa^{+i}) = \chi_{z_n}(\kappa^{+i})$, and
- $t_n \subseteq y$.

Note that because of Claim 2.6 above the y in the definition of E^n_α is uniquely determined by $y \cap \kappa$, and that $y \subseteq z_n$. It is easily seen that E^n_α is a club in κ.

Claim 2.9. (1) *Let $\alpha < \beta < \mu$ then for large enough $n < \omega$, $E^n_\beta \subseteq E^n_\alpha$.*
(2) *If $\alpha < \beta$ are both in S such that α is a limit point of C_β then for every n, $E^n_\beta \subseteq E^n_\alpha$.*

Proof. Pick $\rho < \delta$ such that any subalgebra of $\mathcal{A}_\beta$ containing ρ is closed under the operations of $\mathcal{A}_\alpha$. Let n be large enough such that $\rho \in t_n$. Then for $\xi \in E^n_\beta$ let y be the (unique) witness for this. Then since $t_n \subseteq y$, $\rho \in y$, and then y also witnesses $\xi \in E^n_\alpha$.

The argument for the second clause in the claim is completely analogous. $\square$

In order to deal with the requirement that the cofinality in W of the values of the characteristic function of the required x_n will be at least λ, for every $\xi < \kappa$ with $\mathrm{cf}^W(\xi) < \lambda$, we fix (in W) a club e_ξ in ξ of order type less than λ. Let C be a club in κ, and for $\gamma < \sigma < \kappa$ define the operation $T(C, \gamma, \sigma)$ by induction as follows: If $C \cap [\gamma, \sigma] = \emptyset$, let $T(C, \gamma, \sigma) = \emptyset$. Otherwise, let $\xi = \sup(C \cap [\gamma, \sigma])$. If $\mathrm{cf}(\xi)^W \geq \lambda$, then $T(C, \gamma, \sigma) = \{\xi\}$. If the two previous cases fail let

$$T(C, \gamma, \sigma) :=$$
$$\{\xi\} \cup \bigcup \{T(C, \bar\gamma, \bar\sigma) \mid \bar\gamma, \bar\sigma \text{ are two consecutive memebrs of } (e_\xi \setminus \gamma) \cup \{\gamma\}\}.$$

It is easily seen by induction that $T(C, \gamma, \sigma)$ is a subset of $C \cap [\gamma, \sigma]$ of cardinality less than λ.

Claim 2.10. *Let C be a club in κ and $\gamma < \sigma < \kappa$. Assume that $C \cap [\gamma, \sigma] \cap \mathrm{cof}^W(\geq \lambda) \neq \emptyset$ then $T(C, \gamma, \sigma) \cap \mathrm{cof}^W(\geq \lambda) \neq \emptyset$.*

Proof. By induction on σ. Denote $T = T(C, \gamma, \sigma)$. By assumption we can pick $\zeta \in C \cap [\gamma, \sigma] \cap \mathrm{cof}^W(\geq \lambda)$. Since $\xi := \sup(C \cap [\gamma, \sigma]) \in T$, there is $\rho \in T$, such that $\rho \geq \zeta$. Let ρ be the least such. If $\mathrm{cf}^W(\rho) \geq \lambda$ we are done. Otherwise $\gamma \leq \zeta < \rho$. Then there are two consecutive members of $(e_\xi \setminus \gamma) \cup \{\gamma\}$, $\bar\gamma < \bar\sigma$, such that $\zeta \in [\bar\gamma, \bar\sigma]$. Since $\bar\sigma < \rho \leq \sigma$, by the induction assumption, we get $\bar\zeta \in T(C, \bar\gamma, \bar\sigma)$ with $\mathrm{cf}^W(\bar\zeta) \geq \lambda$. But $T(C, \bar\gamma, \bar\sigma) \subseteq T$, so $\bar\zeta \in T$. $\square$

Claim 2.11. *Let $\langle G_\rho \mid \rho < \lambda \rangle$ be a $\subseteq$-decreasing sequence of clubs of κ, each of them in V. (The sequence does not have to be in V). Let $\gamma < \sigma < \kappa$. Then the sequence $\langle T(G_\rho, \gamma, \sigma) \mid \rho < \lambda \rangle$ is eventually constant.*

Proof. We prove the claim by induction on σ. If for some $\bar\rho < \lambda$, $G_{\bar\rho} \cap [\gamma, \sigma] = \emptyset$, then for $\rho \geq \bar\rho$, $T(G_\rho, \gamma, \sigma) = \emptyset$, and we are done.

So assume otherwise and let $\xi_\rho = \sup(G_\rho \cap [\gamma, \sigma]) = \sup(T(G_\rho, \gamma, \sigma))$. This is a non increasing sequence of ordinals, so without loss of generality we can assume that it is a constant ξ for all ρ. If $\mathrm{cf}^W(\xi) \geq \lambda$, then for every ρ $T(G_\rho, \gamma, \sigma) = \{\xi\}$, and so it is constant. If $\mathrm{cf}^W(\xi) < \lambda$ then for each two successive points in e_ξ ,$\bar\gamma, \bar\sigma$, by the induction assumption $T(G_\rho, \bar\gamma, \bar\sigma)$ is eventually constant. Since $|e_\xi| < \lambda$, we can find $\bar\rho < \lambda$ and $T^*(\bar\gamma, \bar\sigma)$ such that for $\rho \geq \bar\rho$, $T(G_\rho, \bar\gamma, \bar\sigma) = T^*(\bar\gamma, \bar\sigma)$. Then for all $\rho \geq \bar\rho$, by definition, $T(G_\rho, \gamma, \sigma)$ is

$$\{\xi\} \cup \bigcup \{T^*(\bar\gamma, \bar\sigma) \mid \bar\gamma, \bar\sigma \text{are two consecutive memebrs of } (e_\xi \setminus \gamma) \cup \{\gamma\}\}$$

and so we are done. $\square$

Claim 2.12. *Let $E \subseteq G \subseteq H$ be three clubs in κ and $\gamma < \sigma < \kappa$. Assume that $T(H,\gamma,\sigma) \neq T(G,\gamma,\sigma)$ then $T(E,\gamma,\sigma) \neq T(H,\gamma,\sigma)$.*

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Proof. By induction on σ. The only nontrivial case is when $T(E,\gamma,\sigma) \neq \emptyset$. Then clearly $T(G,\gamma,\sigma) \neq \emptyset$ and $T(H,\gamma,\sigma) \neq \emptyset$. So $\xi_E = \sup(E \cap [\gamma,\sigma]) = \sup(T(E,\gamma,\sigma))$,$\xi_G = \sup(E \cap [\gamma,\sigma]) = \sup(T(G,\gamma,\sigma))$, $\xi_H = \sup(H \cap [\gamma,\sigma]) = \sup(T(H,\gamma,\sigma))$ are all defined and $\xi_E \leq \xi_G \leq \xi_H$. If $\xi_G < \xi_H$ then $\xi_E < \xi_H$ and the claim is proved. So assume $\xi_G = \xi_H$. If $\xi_E < \xi_G$ then again the claim is verified.

So assume $\xi = \xi_E = \xi_G = \xi_H$. If $\mathrm{cf}^W(\xi) \geq \lambda$ then $T(H,\gamma,\sigma) = T(G,\gamma,\sigma) = \{\xi\}$, contradiction with the assumption of the claim. Then $\mathrm{cf}^W(\xi) < \lambda$, and so there must be two successive points in e_ξ $\bar{\gamma},\bar{\sigma}$ such that $T(G,\bar{\gamma},\bar{\sigma}) \neq T(H,\bar{\gamma},\bar{\sigma})$. By the inductive hypothesis $T(E,\bar{\gamma},\bar{\sigma}) \neq T(H,\bar{\gamma},\bar{\sigma})$, and so $T(E,\gamma,\sigma) \neq T(H,\gamma,\sigma)$. □

Fix a sequence of order type ω, cofinal in κ , $\langle \kappa_n \mid n < \omega \rangle$. For $\alpha < \mu$ and $n < \omega$, define

$$d^n_\alpha = \bigcup_{k<\omega} T(E^n_\alpha, \kappa_k, \kappa_{k+1}).$$

For $\alpha < \beta < \mu$ we say that there is a *major change* between α and β, if there are infinitely many n's such that $d^n_\alpha \neq d^n_\beta$.

Claim 2.13. *Let $\alpha < \beta < \gamma < \mu$ and suppose that there is a major change between α and β. Then there is a major change between α and γ.*

Proof. By claim 2.9 there is $\bar{n} < \omega$, such that for every $n \geq \bar{n}$, we have that $E^n_\gamma \subseteq E^n_\beta \subseteq E^n_\alpha$. By assumptions the set $R = \{n \mid d^n_\alpha \neq d^n_\beta\}$ is infinite . By claim 2.12 for $n \in R, n \geq \bar{n}$, $d^n_\alpha \neq d^n_\gamma$.

□

Lemma 2.14. *There is $\alpha < \mu$, such that for $\alpha \leq \beta < \gamma$, there is no major change between β and γ.*

Proof. By claim 2.13 it is enough to see that there is $\alpha < \mu$ such for all $\alpha < \beta$ there is no major change between α and β. If this fails, then for every $\alpha < \mu$, let $f(\alpha) > \alpha$ be such that, there is a major change between α and $f(\alpha)$. Let G be the club of the ordinals in μ which are closed under the function f.

Since $S \cap \mathrm{cof}^W(\lambda)$ is stationary in μ, pick $\zeta \in S \cap G$ such that $\mathrm{cf}^W(\zeta) = \lambda$. Let $H = G \cap \lim(C_\zeta)$. Since λ is uncountable, H is a club in ζ of order type λ, and $H \subset S$.

If $\alpha < \beta$ are both in H, then α is a limit point of C_β, and so by claim 2.9, for all $n < \omega$, $E^n_\beta \subseteq E^n_\alpha$. Then for all $n < \omega$, the sequence $\langle E^n_\alpha \mid \alpha \in H \rangle$ is a decreasing sequence of λ-many clubs in κ. Hence by applying claim 2.11 ω-many times, for each n, we get that there is $\alpha \in H$, such that for all n and all $\beta \in H, \alpha < \beta$, $d^n_\alpha = d^n_\beta$. By definition of G, $\alpha < f(\alpha) < \beta$. Then there is major change between α and $f(\alpha)$, but there is no change at all between α and β, which contradicts claim 2.13. $\square$

Fix $\alpha < \mu$, such that there is no major change after α and define $d^n = d^n_\alpha$. For every $n < \omega$, as a club in κ, E^n_α contains unboundedly many ordinals of cofinality $\geq \lambda$. Then, by claim 2.10, for every $n < \omega$, d^n also contains unboundedly many ordinals of cofinality $\geq \lambda$. Hence we can define an increasing sequence of ordinals cofinal in κ, $\langle \rho_n \mid n < \omega \rangle$, such that for all n, $\rho_n \in d^n$ and $\mathrm{cf}^W(\rho_n) \geq \lambda$.

Claim 2.15. *For every $\beta < \mu$, for large enough n, $\rho_n \in E^n_\beta$.*

Proof. If $\beta < \alpha$, then for large enough n $E^n_\alpha \subseteq E^n_\beta$, so for large enough n, $d^n \subseteq E^n_\alpha \subseteq E^n_\beta$. If $\alpha \leq \beta$, then by definition of α, for large enough n, $d^n_\beta = d^n_\alpha = d^n$, and so again $d^n \subseteq E^n_\beta$. This implies that for for every $\beta < \mu$, for large enough n, $\rho_n \in E^n_\beta$. $\square$

We can now finish the proof of the theorem. For $n < \omega$, let x_n be the unique member of $P_\kappa(\delta)$ such that $x_n \in D$, for all $0 < i \leq m+1$ $\sup(x_n \cap \kappa^{+i}) = \sup(z_n \cap \kappa^{+i})$ and $x_n \cap \kappa = \rho_n$. Such x_n exists since $\rho_n \in E^n_\alpha$. Also note that automatically we get $t_n \subseteq x_n$.

We claim that the sequence $\langle x_n | n < \omega \rangle$ is as desired. Let $\beta < \mu$, and let n be large enough, so that $\rho_n \in E^n_\beta$. Then by definition of E^n_β, $x_n \in D_\beta$. $\square$

3. Consequences on square properties

Theorem 3.1. *Suppose $V \subset W$ and $\kappa < \nu$ are such that, ν is a regular cardinal in V, κ is a singular cardinal in W of countable cofinality, for all V-regular cardinals τ, with $\kappa \leq \tau \leq \nu$, $W \models \mathrm{cf}(\tau) = \omega$, and $(\nu^+)^V = (\kappa^+)^W$. Then in W, $\square_{\kappa,\omega}$ holds.*

Proof. Work in V. Denote $\mu := \nu^+$. Let χ be some big enough cardinal, and $<_\chi$ be a well-ordering of H_χ. For $\alpha < \mu$ with $\mathrm{cf}(\alpha) < \kappa$, let $\langle M^\alpha_\delta \mid \delta < \nu \rangle$ be a continuous $\subset$-increasing sequence of elementary submodels of $\langle H_\chi, \in, <_\chi \ldots \rangle$, such that:

(1) $\alpha, \kappa \in M^\alpha_0$,
(2) for each $\delta < \nu$, $|M^\alpha_\delta| < \nu$,
(3) for each $\delta < \nu$, $M^\alpha_\delta \cap \nu \in \nu$.

Claim 3.2. *For all $\alpha < \mu$, $\delta < \nu$ with $\mathrm{cf}(\delta) > \omega$, if $M^\alpha_\delta \cap \alpha$ is cofinal in α, then $M^\alpha_\delta \cap \alpha$ is ω-closed.*

Proof. Suppose for contradiction that $\langle \beta_i \mid i < \omega \rangle$ is an increasing sequence of points in $M^\alpha_\delta \cap \alpha$, such that $\beta := \sup_i \beta_i \notin M^\alpha_\delta \cap \alpha$. Let β^* be the least ordinal in $M^\alpha_\delta \cap \alpha$ above β. If $\mathrm{cf}(\beta^*) = \lambda < \nu$, then $\lambda \in M^\alpha_\delta \cap \nu \in \nu$. Then $\lambda + 1 \subset M^\alpha_\delta$, and so M^α_δ is cofinal in β^*, which is a contradiction.

It follows that the cofinality of β^* must be ν. Say $M^\alpha_\delta \models h : \nu \to \beta^*$ is cofinal. For every i, let $\beta^*_i \in \mathrm{ran}(h) \setminus \beta_i$. The order type of $\mathrm{ran}(h)$ is $\nu \cap M^\alpha_\delta$.

And $\mathrm{cf}(\nu \cap M^\alpha_\delta) = \mathrm{cf}(\delta) > \omega$. So there is something in the range of h above β. Contradiction with the choice of β^*.

$\square$

Claim 3.3. *Let $\alpha < \mu$. $D_\alpha := \{\sup(M^\alpha_\delta \cap \nu) \mid \delta < \nu\}$ is club in ν.*

Proof. D_α is closed by construction since the sequence is continuous. To see that it is unbounded, note that $\nu \subset \bigcup_\delta M^\alpha_\delta$. Note that this also implies that $\alpha \subset \bigcup_\delta M^\alpha_\delta$, since the $<_\chi$-least function in H_χ from ν onto α is in each M^α_δ. $\square$

By Theorem 2.2 there is a sequence $\langle \lambda_i \mid i < \omega\rangle \in W$, such that for all large i, $\lambda_i \in \overline{D_\alpha}$ and for all i, $\mathrm{cf}^W(\lambda_i) > \omega$. For $\alpha < \mu$ with $\mathrm{cf}^V(\alpha) < \kappa$, define $\mathcal{C}_\alpha := \{\overline{M^\beta_\delta \cap \alpha} \mid \alpha \leq \beta < \mu, \delta < \nu, M^\beta_\delta$ is cofinal in β and $\alpha, (\exists i) M^\beta_\delta \cap (\nu)^V = \lambda_i\}$. Here $\overline{M^\beta_\delta \cap \alpha}$ denotes the closure of $M^\beta_\delta \cap \alpha$. Otherwise, $\mathrm{cf}^W(\alpha) = \omega$ and we let $\mathcal{C}_\alpha$ be a singleton of some ω-sequence cofinal in α.

Claim 3.4. $1 \leq |\mathcal{C}_\alpha| \leq \omega$.

Proof. Each $\mathcal{C}_\alpha$ is nonempty because for all large i, $\lambda_i \in D_\alpha$. Now suppose $\beta, \beta', \delta, \delta'$ are such that:

- $M^\beta_\delta \cap \nu = M^{\beta'}_{\delta'} \cap \nu = \lambda_i$,
- $M^\beta_\delta, M^{\beta'}_{\delta'}$ are both cofinal in α.

We want to show that $M^\beta_\delta \cap \alpha = M^{\beta'}_{\delta'} \cap \alpha$. If $\mathrm{cf}(\alpha) = \omega$, then $\alpha \in M^\beta_\delta, M^{\beta'}_{\delta'}$. By elementarity, the $\prec_\chi$-least function from ν onto α is in $M^\beta_\delta \cap \nu = M^{\beta'}_{\delta'} \cap \nu$, and the result follows. Now suppose that $\mathrm{cf}(\alpha) > \omega$. We have $\mathrm{cf}(\delta) = \mathrm{cf}(\delta') = \mathrm{cf}(\lambda_i) > \omega$. Then $M^\beta_\delta \cap \alpha, M^{\beta'}_{\delta'} \cap \alpha$ are both ω-clubs in α, and so $M^\beta_\delta \cap M^{\beta'}_{\delta'} \cap \alpha$ is an ω-club in α. By the above for any $\eta \in M^\beta_\delta \cap M^{\beta'}_{\delta'} \cap \alpha$, $M^\beta_\delta \cap \eta = M^{\beta'}_{\delta'} \cap \eta$, and so $M^\beta_\delta \cap \alpha = M^{\beta'}_{\delta'} \cap \alpha$.

$\square$

Next we use $\langle \mathcal{C}_\alpha \mid \alpha < \mu\rangle$ to obtain a $\square_{\kappa,\omega}$ sequence. Since ν has cardinality κ in W, there is a sequence $\langle F_\beta \mid \beta < \nu\rangle$, such that each F_β is a club in β of order type at most κ, and for $\delta \in \lim(F_\beta)$, $F_\beta \cap \delta = F_\delta$. This is similar to arguments in Section 6.1 of [1].

Now enumerate $\mathcal{C}_\alpha := \{C^\alpha_n \mid n < \omega\}$ and let each $C^\alpha_n = \{\gamma^{\alpha,n}_\xi \mid \xi < \nu^\alpha_n\}$, where $\nu^\alpha_n = o.t.(C^\alpha_n) < \nu$. Define $E^\alpha_n := \{\gamma^{\alpha,n}_\xi \mid \xi \in F_{\gamma^\alpha_n}\}$ and let $\mathcal{E}_\alpha = \{E^\alpha_n \mid n < \omega\}$. It is routine to verify that $\langle \mathcal{E}_\alpha \mid \alpha < \mu\rangle$ is a $\square_{\kappa,\omega}$ sequence.

$\square$

We conclude with some open question.

Question 1. *Can we replace κ^{+m} in Theorem 2.7 with any (possibly singular) V-cardinal ν?*

We remark that by [5], we cannot take ν to be singular in Theorem 3.1.

Question 2. *Can we get a similar result as in Theorem 3.1 for uncountable cofinality?*

The main obstacle here would be to deal with the points whose V-cofinality is κ when trying to build a square sequence.

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